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The Complete Asymptotic Expansion for the Degree of Approximation of Lipschitz Functions by Hermite–Fejér Interpolation Polynomials

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The complete asymptotic expansion is derived for the degree of approximation of Lipschitz functions by Hermite–Fejér interpolation polynomials based on the zeros of the Chebyshev polynomials of the first kind.

1. INTRODUCTION

Let f be a real-valued function on $[-1, 1]$ and, for $k = 1, 2, 3, \ldots, n$, denote by

$$x_k := \cos \frac{(2k-1)\pi}{2n} \tag{1.1}$$

the zeros of the Chebyshev polynomial of the first kind

$$T_n(x) := \cos n\theta, \qquad \text{where } -1 \leqslant x := \cos\theta \leqslant 1. \tag{1.2}$$

Then there exists a unique polynomial $H_{2n-1}(f; x)$ of degree not exceeding $2n-1$ such that

$$H_{2n-1}(f; x_k) = f(x_k) \qquad \text{for} \quad k = 1, 2, 3, \ldots, n \tag{1.3}$$

and

$$H'_{2n-1}(f; x_k) = 0 \qquad \text{for} \quad k = 1, 2, 3, \ldots, n. \tag{1.4}$$

$H_{2n-1}(f; x)$ is called the Hermite–Fejér interpolation polynomial associated with f and based on the zeros of $T_n(x)$. These polynomials were employed by Fejér [3] in 1916 in presenting a new proof of the Weierstrass approximation theorem.

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THEOREM 1 (L. Fejér). *If* $f \in \mathscr{C}[-1, 1]$, *then*

$$\lim_{n \to \infty} \|H_{2n-1}(f) - f\|_\infty = 0,$$

where $\|\cdot\|_\infty$ *denotes the uniform norm on* $\mathscr{C}[-1, 1]$.

In 1950, the first estimate of the rate of convergence of Hermite–Fejér interpolation polynomials was provided by Popoviciu [8]. This estimate was framed in terms of the modulus of continuity of f, which is defined as follows.

$$\omega(f; \delta) := \sup\{|f(x) - f(y)| : x, y \in [-1, 1], |x - y| \leqslant \delta\} \quad \text{for all} \quad \delta \geqslant 0.$$

THEOREM 2 (T. Popoviciu). *If* $f \in \mathscr{C}[-1, 1]$, *then, for* $n = 1, 2, 3, \ldots$,

$$\|H_{2n-1}(f) - f\|_\infty \leqslant 2\omega(f; 1/\sqrt{n}).$$

This result was improved by Moldovan [7] (1954) and later, from a different point of view, by Shisha and Mond [10] (1965).

THEOREM 3 (E. Moldovan, O. Shisha, and B. Mond). *If* $f \in \mathscr{C}[-1, 1]$, *then, for* $n = 4, 5, 6, \ldots$,

$$\|H_{2n-1}(f) - f\|_\infty < C_1 \omega\left(f; \frac{\log n}{n}\right),$$

where C_1 *is an absolute positive constant.*

The next major improvement in estimating $\|H_{2n-1}(f) - f\|_\infty$ came from Bojanic [1] in 1969. To understand his result, we shall need the following:

DEFINITION. Let $\Omega(\delta)$ be an increasing, subadditive, continuous function defined on $\mathbb{R}^+$ with $\Omega(0) = 0$ and let $M > 0$. Then we define

$$\mathscr{C}_M(\Omega) := \{f \in \mathscr{C}[-1, 1] : \omega(f; \delta) \leqslant M\Omega(\delta) \text{ for all } \delta \geqslant 0\}.$$

THEOREM 4 (R. Bojanic). *There exist positive constants* C_2 *and* C_3 *such that for* $n = 2, 3, 4, \ldots$,

$$\frac{C_2 M}{n} \sum_{k=2}^{n} \Omega\left(\frac{1}{k}\right) \leqslant \sup\{\|H_{2n-1}(f) - f\|_\infty : f \in \mathscr{C}_M(\Omega)\} \leqslant \frac{C_3 M}{n} \sum_{k=1}^{n} \Omega\left(\frac{1}{k}\right).$$

The aim of this paper is to estimate the quantity

$$\Delta_n := \sup\{\|H_{2n-1}(f) - f\|_\infty : f \in \text{Lip } 1\},$$

where $f \in \mathrm{Lip}\, 1$ means that

$$|f(x)-f(y)| \leqslant |x-y| \qquad \text{for all } x, y \text{ in } [-1, 1].$$

Indeed, we shall call Δ_n the *degree of approximation* of continuous real-valued functions of class Lip 1 by Hermite–Fejér interpolation polynomials and we shall obtain the complete asymptotic expansion for Δ_n.

Note that when $\Omega(\delta) := \delta$, Bojanic's theorem gives the *order of the first term* of the complete asymptotic expansion for Δ_n as $\log n/n$. This is in agreement with the first term of our asymptotic expansion for Δ_n, which appears below.

THEOREM 5.

$$\Delta_n \sim \frac{2}{\pi}\frac{\log n}{n} + \frac{2}{\pi}\left[\log\frac{8}{\pi} + \gamma\right]\frac{1}{n} + 4\sum_{k=1}^{\infty}\frac{A_k^*}{n^{2k+1}} \qquad as \quad n = 2m \to \infty,$$

where

$$A_k^* = \frac{(-1)^{k-1}(2^{2k-1}-1)^2\, B_{2k}^2}{2k(2k)!}\left(\frac{\pi}{2}\right)^{2k-1} \qquad for\ k = 1, 2, 3, ...,$$

γ is Euler's constant and B_{2k} for $k = 1, 2, 3, ...,$ represent Bernoulli numbers.

2. TECHNICAL PRELIMINARIES

We first recall that the formula for $H_{2n-1}(f; x)$ is given by

$$H_{2n-1}(f; x) = \sum_{k=1}^{n} f(x_k)\, h_k(x), \tag{2.1}$$

where

$$h_k(x) = \frac{1}{n^2}(1 - xx_k)\left(\frac{T_n(x)}{x - x_k}\right)^2 \tag{2.2}$$

and

$$x_k = x_k^{(n)} = \cos\frac{(2k-1)\pi}{2n} \qquad \text{for} \quad k = 1, 2, 3, ..., n.$$

From (1.3), (1.4) and the uniqueness of $H_{2n-1}(f; x)$, we have

$$H_{2n-1}(1; x) = \sum_{k=1}^{n} h_k(x) \equiv 1. \tag{2.3}$$

For $-1 \leqslant x \leqslant 1$, define

$$\Delta_n(x) := \sup\{|H_{2n-1}(f; x) - f(x)| : f \in \text{Lip } 1\}$$

and

$$\phi_x(t) := |x - t| \qquad \text{for all} \quad t \in [-1, 1].$$

Now, on the one hand, $\phi_x \in \text{Lip } 1$ and hence

$$\Delta_n(x) \geqslant |H_{2n-1}(\phi_x; x) - \phi_x(x)| = \sum_{k=1}^{n} |x - x_k|\, h_k(x).$$

On the other hand, for any $f \in \text{Lip } 1$, we have by (2.1) and (2.3),

$$\begin{aligned} |H_{2n-1}(f; x) - f(x)| &= \left| \sum_{k=1}^{n} [f(x_k) - f(x)]\, h_k(x) \right| \\ &\leqslant \sum_{k=1}^{n} |x - x_k|\, h_k(x). \end{aligned}$$

It follows that

$$\Delta_n(x) = \sum_{k=1}^{n} |x - x_k|\, h_k(x). \tag{2.4}$$

For notational convenience, we now put $x_0 := 1$ and $x_{n+1} := -1$. Then some fundamental properties of the function $\Delta_n(x)$ are provided by:

THEOREM 6 (T. Mills). *$\Delta_n(x)$ is a piecewise polynomial on $[-1, 1]$. More precisely, the restriction of $\Delta_n(x)$ to $[x_{j+1}, x_j]$ is a polynomial of degree $2n$ for $j = 1, 2, \ldots, n-1$, whilst the restriction of $\Delta_n(x)$ to $[-1, x_n]$ (resp. $[x_1, 1]$) is a polynomial of degree $2n-1$. Furthermore, $\Delta_n(x)$ has one and only one local maximum in $[x_{j+1}, x_j]$, for $j = 0, 1, 2, \ldots, n$.*

Proof. Now $\Delta_n(x) = \sum_{k=1}^{n} |x - x_k|\, h_k(x)$ for all $x \in [-1, 1]$, where

$$\begin{aligned} h_k(x) &:= \frac{1}{n^2}(1 - x x_k)\left(\frac{T_n(x)}{x - x_k}\right)^2 \\ &= \frac{2^{2n-2}}{n^2}(1 - x x_k)(x - x_1)^2 \cdots (x - x_{k-1})^2 (x - x_{k+1})^2 \cdots (x - x_n)^2 \end{aligned}$$

is a polynomial of degree $2n-1$. Suppose $x \in [x_{j+1}, x_j]$, where $j \in \{1, 2,..., n-1\}$. Then $\Delta_n(x) = \sum_{k=1}^{j} (x_k - x)\, h_k(x) + \sum_{k=j+1}^{n} (x - x_k)\, h_k(x)$ is a polynomial of degree not exceeding $2n$. We now show that the coefficient of x^{2n}—call it a_{2n}—is nonzero.

$$\begin{aligned} a_{2n} &= \frac{2^{2n-2}}{n^2} \sum_{k=1}^{j} x_k - \frac{2^{2n-2}}{n^2} \sum_{k=j+1}^{n} x_k \\ &= \frac{2^{2n-2}}{n^2} \left[\sum_{k=1}^{n} x_k - 2 \sum_{k=j+1}^{n} x_k \right] \\ &= \frac{-2^{2n-1}}{n^2} \sum_{k=j+1}^{n} x_k > 0, \end{aligned} \tag{2.5}$$

and so the restriction of $\Delta_n(x)$ to $[x_{j+1}, x_j]$ is a polynomial of degree $2n$. If $x \in [-1, x_n]$, then

$$\begin{aligned} \Delta_n(x) &= \sum_{k=1}^{n} (x_k - x)\, h_k(x) \\ &= \sum_{k=1}^{n} x_k h_k(x) - x \qquad \text{is a polynomial of degree } 2n-1. \end{aligned}$$

Similarly, if $x \in [x_1, 1]$, we can show that $\Delta_n(x)$ is a polynomial of degree $2n-1$. We now show that $\Delta_n(x)$ has one and only one local maximum in $[x_{j+1}, x_j]$, for $j = 0, 1,..., n$.

First, suppose $j \in \{1, 2,..., n-1\}$. Put $\Delta_{n,j}(x) := \sum_{k=1}^{j} (x_k - x)\, h_k(x) + \sum_{k=j+1}^{n} (x - x_k)\, h_k(x)$ for all $x \in \mathbb{R}$. Then $\Delta_{n,j}(x)$ is the polynomial of degree $2n$ which agrees with $\Delta_n(x)$ on $[x_{j+1}, x_j]$. We now establish that $\Delta_{n,j}(x)$ has $2n$ zeros. Observe that

$$\begin{aligned} \Delta'_{n,j}(x) = & \sum_{k=1}^{j} (x_k - x)\, h'_k(x) - \sum_{k=1}^{j} h_k(x) \\ & + \sum_{k=j+1}^{n} (x - x_k)\, h'_k(x) + \sum_{k=j+1}^{n} h_k(x). \end{aligned}$$

But $h_k(x_i) = \delta_{ki}$ and $h'_k(x_i) = 0$ (see, e.g., Rivlin [9, p. 24]), so it follows that

$$\Delta_{n,j}(x_i) = 0 \qquad \text{for} \quad i = 1, 2,..., n$$

and

$$\begin{aligned} \Delta'_{n,j}(x_i) &= -1 \qquad \text{for} \quad i \leqslant j \\ &= 1 \qquad \text{for} \quad i > j. \end{aligned}$$

In this case, $\Delta_{n,j}(x)$ must also vanish in (x_n, x_{n-1}), (x_{n-1}, x_{n-2}),..., (x_{j+2}, x_{j+1}), (x_j, x_{j-1}),..., (x_2, x_1) as well as at $x_1, x_2,..., x_n$. In other words,

$\Delta_{n,j}(x)$ has at least $2n-2$ zeros. We now show that $\Delta_{n,j}(x)$ has a zero in $(-\infty, x_n)$ and a zero in (x_1, ∞). Given $j \in \{1, 2, ..., n-1\}$, it follows from (2.5) that the coefficient of x^{2n} is positive. Hence

$$\lim_{x \to \pm\infty} \Delta_{n,j}(x) = \infty. \tag{2.6}$$

But $\Delta'_{n,j}(x_n) = 1$, $\Delta'_{n,j}(x_1) = -1$ and $\Delta_{n,j}(x)$ is a polynomial, so it follows from (2.6) that $\Delta_{n,j}(x)$ must have a zero in $(-\infty, x_n)$ and another in (x_1, ∞). We deduce that for $j = 1, 2, ..., n-1$, $\Delta_{n,j}(x)$ has $2n$ zeros.

By Rolle's theorem, $\Delta'_{n,j}(x)$ has $2n-1$ zeros. Now if $\Delta_{n,j}(x)$ has 2 local maxima in $[x_{j+1}, x_j]$, then $\Delta'_{n,j}(x)$ would have 3, rather than 1, zero(s) in $[x_{j+1}, x_j]$, which means that $\Delta'_{n,j}(x)$ would have $2n+1$ zeros. But this contradicts the fact that $\Delta'_{n,j}(x)$ is a polynomial of degree $2n-1$. Thus for $j \in \{1, 2, ..., n-1\}$, $\Delta_{n,j}(x)$ has one and only one local maximum in $[x_{j+1}, x_j]$. For $j = 0$ we see that

$$\Delta_{n,0}(x) := \sum_{k=1}^{n} (x - x_k)\, h_k(x) = x - \sum_{k=1}^{n} x_k h_k(x)$$

$$\text{is a polynomial of degree } 2n-1. \tag{2.7}$$

Also $\Delta_{n,0}(x_i) = 0$ for $i = 1, 2, 3, ..., n$ and $\Delta'_{n,0}(x_i) = 1$ for $i = 1, 2, 3, ..., n$. But this means that $\Delta_{n,0}(x)$ has at least $2n-1$ zeros. From (2.7) and Rolle's theorem, we deduce that $\Delta_{n,0}(x)$ is monotonically increasing in $[x_1, 1]$ and so $\max\{\Delta_n(x) : x_1 \leqslant x \leqslant 1\} = \Delta_n(1)$. The case $j = n$ may be treated similarly. This concludes the proof of Theorem 6.

Now that we have obtained an explicit formulation for $\Delta_n(x)$ (see (2.4) for details), we shall need to determine the point(s) at which $\Delta_n(x)$ attains its maximum value. For even values of n, the answer to this problem is provided by the following theorem.

THEOREM 7. *For* $n = 2, 4, 6, ...,$

$$\Delta_n(x) \leqslant \Delta_n(0) \qquad \textit{for all} \quad x \in [-1, 1].$$

The proof of this theorem is both difficult and somewhat delicate. Accordingly, we shall provide an outline of the proof later in this article.

Assuming the validity of Theorem 7, we see that for even values of n,

$$\begin{aligned} \Delta_n &:= \sup\{\Delta_n(x) : -1 \leqslant x \leqslant 1\} \\ &= \max\left\{\sum_{k=1}^{n} |x - x_k|\, h_k(x) : -1 \leqslant x \leqslant 1\right\} \qquad \text{from (2.4)} \\ &= \frac{2}{n^2} \sum_{k=1}^{m} \frac{1}{x_k}, \qquad \text{where } n = 2m. \end{aligned} \tag{2.8}$$

Formula (2.8) is of fundamental importance in enabling us to obtain the complete asymptotic expansion for Δ_n.

Furthermore, we shall need the following results in the proof of Theorem 5.

LEMMA 1.

$$\sum_{p=2}^{\infty} ((-1)^{p-1}(2^{2p-1}-1)\, B_{2p}/p(2p)!)(\pi/2)^{2p-1} = (2/\pi)\log(4/\pi) - (\pi/24).$$

Proof. From [4, p. 35], we have

$$\operatorname{cosec}\theta = \sum_{p=0}^{\infty} \frac{(-1)^{p-1}2(2^{2p-1}-1)B_{2p}}{(2p)!}\,\theta^{2p-1},$$

valid for $\theta^2 < \pi^2$, where $B_0 = 1$, $B_2 = \frac{1}{6}$, $B_4 = \frac{-1}{30}$,..., represent Bernoulli numbers. That is, $\operatorname{cosec}\theta = (1/\theta) + (\theta/6) + \sum_{p=2}^{\infty}((-1)^{p-1}2(2^{2p-1}-1)B_{2p}/(2p)!)\,\theta^{2p-1}$. Integrating both sides of the above equation with respect to θ, we have

$$\log\left(2\tan\frac{\theta}{2}\right) = \log\theta + \frac{\theta^2}{12} + \theta\sum_{p=2}^{\infty}\frac{(-1)^{p-1}(2^{2p-1}-1)B_{2p}}{p(2p)!}\,\theta^{2p-1}.$$

But then

$$\sum_{p=2}^{\infty}\frac{(-1)^{p-1}(2^{2p-1}-1)\,B_{2p}}{p(2p)!}\,\theta^{2p-1} = \frac{1}{\theta}\log\left(\frac{\tan\theta/2}{\theta/2}\right) - \frac{\theta}{12}.$$

In particular, when $\theta = \pi/2$ we have

$$\sum_{p=2}^{\infty}\frac{(-1)^{p-1}(2^{2p-1}-1)\,B_{2p}}{p(2p)!}\left(\frac{\pi}{2}\right)^{2p-1} = \frac{2}{\pi}\log\frac{4}{\pi} - \frac{\pi}{24}$$

and the lemma is proved.

Now put $A(j) := (-1)^j 2(2^{2j+1}-1)\,B_{2j+2}/(2j+2)!$ for $j = -1, 0, 1,\ldots,$ and $P_r^m := m!/(m-r)!$. Whilst Lemma 2 will seem somewhat obscure at this stage, it is nevertheless vitally important in the proof of Theorem 5.

LEMMA 2. *For $k = 1, 2, 3,\ldots,$*

$$\sum_{j=k}^{\infty}\binom{2j+1}{2k-1} A(j)\left(\frac{\pi}{2}\right)^{2j+1} = \frac{2}{\pi} - \left(\frac{\pi}{2}\right)^{2k-1} A(k-1).$$

Proof. By induction, we can show that

$$\frac{d^{2k-1}}{d\theta^{2k-1}}(\operatorname{cosec}\theta)+\frac{(2k-1)!\,A(-1)}{\theta^{2k}}-(2k-1)!\,A(k-1)$$

$$=\sum_{j=k}^{\infty} P_{2k-1}^{2j+1}\,A(j)\,\theta^{2j-2k+2} \qquad \text{for} \quad k=1, 2, 3,\ldots.$$

Now multiply both sides by $\theta^{2k-1}/(2k-1)!$. We have

$$\frac{\theta^{2k-1}}{(2k-1)!}\frac{d^{2k-1}}{d\theta^{2k-1}}(\operatorname{cosec}\theta)+\frac{A(-1)}{\theta}-\theta^{2k-1}A(k-1)$$

$$=\sum_{j=k}^{\infty}\binom{2j+1}{2k-1}A(j)\,\theta^{2j+1}.$$

Replacing θ by $\pi/2$ in the above equation, we have

$$\sum_{j=k}^{\infty}\binom{2j+1}{2k-1}A(j)\left(\frac{\pi}{2}\right)^{2j+1}=\frac{2}{\pi}-\left(\frac{\pi}{2}\right)^{2k-1}A(k-1)$$

and the lemma is proved.

We now have the necessary machinery for the proof of Theorem 5.

3. Proof of Theorem 5

From (2.8) we have

$$\begin{aligned}
\Delta_n &:= \sup\{\|H_{2n-1}(f)-f\|_\infty : f\in \text{Lip } 1\}\\
&=\frac{2}{n^2}\sum_{k=1}^{m}\frac{1}{x_k} \qquad \text{for} \quad n=2m \text{ even}\\
&=\frac{2}{n^2}\sum_{j=1}^{m}\operatorname{cosec}\theta_j, \qquad \text{where } x=\cos\theta\\
&=\frac{2}{n^2}\sum_{p=0}^{\infty}\frac{(-1)^{p-1}2(2^{2p-1}-1)\,B_{2p}}{(2p)!}\sum_{j=1}^{m}\frac{(2j-1)^{2p-1}\pi^{2p-1}}{2^{2p-1}n^{2p-1}}\\
&=\sum_{p=0}^{\infty}\frac{(-1)^{p-1}4(2^{2p-1}-1)\,B_{2p}}{(2p)!}\left(\frac{\pi}{2}\right)^{2p-1}\frac{1}{n^{2p+1}}\sum_{j=1}^{m}(2j-1)^{2p-1}\\
&=\sum_{p=0}^{\infty}U_p, \qquad \text{say}\\
&=U_0+U_1+\sum_{p=2}^{\infty}U_p. \qquad (3.1)
\end{aligned}$$

We now evaluate U_0 and U_1. First,

$$U_0=(-1)\,4\left(\frac{1}{2}-1\right)B_0\left(\frac{2}{\pi}\right)\frac{1}{n}\sum_{j=1}^{m}\frac{1}{2j-1}$$

$$=\frac{4}{\pi n}\sum_{j=1}^{m}\frac{1}{2j-1}$$

$$\sim\frac{4}{\pi n}\left[\frac{1}{2}\log m+\log 2+\frac{\gamma}{2}+\sum_{k=1}^{\infty}\frac{B_{2k}}{2k}\frac{(2^{2k-1}-1)}{n^{2k}}\right] \qquad \text{by [4, p. 3]},$$

where $\gamma=0.5772...$ is Euler's constant. Thus

$$U_0\sim\frac{4}{\pi n}\left[\frac{1}{2}\log n+\frac{1}{2}\log 2+\frac{\gamma}{2}+\sum_{k=1}^{\infty}\frac{B_{2k}}{2k}(2^{2k-1}-1)\cdot\frac{1}{n^{2k}}\right]$$

$$\sim\frac{2}{\pi}\frac{\log n}{n}+\frac{2}{\pi}\frac{\log 2}{n}+\frac{2\gamma}{\pi n}+\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{B_{2k}}{2k}(2^{2k-1}-1)\cdot\frac{1}{n^{2k+1}} \tag{3.2}$$

Second,

$$U_1=\frac{4B_2}{2!}\cdot\frac{\pi}{2}\cdot\frac{1}{n^3}\sum_{j=1}^{m}(2j-1)$$

$$=\frac{\pi}{24}\cdot\frac{4m^2}{n^3}$$

$$=\frac{\pi}{24n}. \tag{3.3}$$

In order to evaluate U_p for $p=2, 3, 4,...,$ we shall first need to consider $\sum_{j=1}^{m}(2j-1)^{2p-1}$. Now

$$\sum_{j=1}^{m}(2j-1)^{2p-1}$$

$$=\frac{2^{2p-1}}{2p}m^{2p}-\sum_{r=1}^{p-1}\frac{1}{2r}\binom{2p-1}{2r-1}2^{2(p-r)}(2^{2r-1}-1)\,B_{2r}m^{2(p-r)} \qquad \text{by [4, p. 2]}$$

$$=\frac{n^{2p}}{4p}-\sum_{r=1}^{p-1}\frac{1}{2r}\binom{2p-1}{2r-1}(2^{2r-1}-1)\,B_{2r}n^{2p-2r}.$$

It follows that

$$U_p=\frac{(-1)^{p-1}4(2^{2p-1}-1)\,B_{2p}}{(2p)!}\left(\frac{\pi}{2}\right)^{2p-1}\frac{1}{n^{2p+1}}$$

$$\times\left[\frac{n^{2p}}{4p}-\sum_{r=1}^{p-1}\frac{1}{2r}\binom{2p-1}{2r-1}(2^{2r-1}-1)\,B_{2r}n^{2p-2r}\right]$$

$$=\frac{1}{n}\frac{(-1)^{p-1}(2^{2p-1}-1)\,B_{2p}}{p(2p)!}\left(\frac{\pi}{2}\right)^{2p-1}$$

$$-\frac{(-1)^{p-1}4(2^{2p-1}-1)\,B_{2p}}{(2p)!}\left(\frac{\pi}{2}\right)^{2p-1}$$

$$\times\sum_{r=1}^{p-1}\frac{1}{2r}\binom{2p-1}{2r-1}(2^{2r-1}-1)\,B_{2r}\cdot\frac{1}{n^{2r+1}}$$

and so

$$\sum_{p=2}^{\infty}U_p\sim\frac{1}{n}\sum_{p=2}^{\infty}\frac{(-1)^{p-1}(2^{p-1}-1)\,B_{2p}}{p(2p)!}\left(\frac{\pi}{2}\right)^{2p-1}$$

$$-4\sum_{p=2}^{\infty}\frac{(-1)^{p-1}(2^{2p-1}-1)\,B_{2p}}{(2p)!}\left(\frac{\pi}{2}\right)^{2p-1}$$

$$\times\left[\sum_{r=1}^{p-1}\frac{1}{2r}\binom{2p-1}{2r-1}(2^{2r-1}-1)\,B_{2r}\cdot\frac{1}{n^{2r+1}}\right]$$

$$\sim\frac{1}{n}\left[\frac{2}{\pi}\log\frac{4}{\pi}-\frac{\pi}{24}\right]$$

$$-4\sum_{p=2}^{\infty}\frac{(-1)^{p-1}(2^{2p-1}-1)\,B_{2p}}{(2p)!}\left(\frac{\pi}{2}\right)^{2p-1}$$

$$\times\left[\sum_{r=1}^{p-1}\frac{1}{2r}\binom{2p-1}{2r-1}(2^{2r-1}-1)\,B_{2r}\cdot\frac{1}{n^{2r+1}}\right]$$

by Lemma 1

$$\sim\frac{1}{n}\left[\frac{2}{\pi}\log\frac{4}{\pi}-\frac{\pi}{24}\right]$$

$$-4\sum_{k=1}^{\infty}\frac{(-1)^{k}(2^{2k+1}-1)\,B_{2k+2}}{(2k+2)!}\left(\frac{\pi}{2}\right)^{2k+1}$$

$$\times\left[\sum_{j=1}^{k}\frac{1}{2j}\binom{2k+1}{2j-1}(2^{2j-1}-1)\,B_{2j}\cdot\frac{1}{n^{2j+1}}\right]$$

$$\sim\frac{1}{n}\left[\frac{2}{\pi}\log\frac{4}{\pi}-\frac{\pi}{24}\right]$$

$$-4\sum_{k=1}^{\infty}\frac{(2^{2k-1}-1)\,B_{2k}}{2k}$$

$$\times\left[\sum_{j=k}^{\infty}\frac{(-1)^{j}(2^{j+1}-1)\,B_{2j+2}}{(2j+2)!}\binom{2j+1}{2k-1}\left(\frac{\pi}{2}\right)^{2j+1}\right]\frac{1}{n^{2k+1}}$$

$$\sim \frac{1}{n}\left[\frac{2}{\pi}\log\frac{4}{\pi}-\frac{\pi}{24}\right]$$

$$-4\sum_{k=1}^{\infty}\frac{(2^{2k-1}-1)\,B_{2k}}{2k}$$

$$\times\left[\frac{1}{\pi}-\frac{(-1)^{k-1}(2^{2k-1}-1)\,B_{2k}}{(2k)!}\left(\frac{\pi}{2}\right)^{2k-1}\right]\frac{1}{n^{2k+1}} \qquad \text{by Lemma 2}$$

$$\sim\frac{2}{\pi n}\log\frac{4}{\pi}-\frac{\pi}{24n}-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(2^{2k-1}-1)\,B_{2k}}{2k}\cdot\frac{1}{n^{2k+1}}$$

$$+4\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(2^{2k-1}-1)^2\,B_{2k}^2}{2k(2k)!}\left(\frac{\pi}{2}\right)^{2k-1}\cdot\frac{1}{n^{2k+1}}. \tag{3.4}$$

From (3.1), (3.2), (3.3), and (3.4) we deduce that

$$\Delta_n = U_0 + U_1 + \sum_{p=2}^{\infty} U_p$$

$$\sim\frac{2}{\pi}\frac{\log n}{n}+\frac{2}{\pi}\frac{\log 2}{n}+\frac{2\gamma}{\pi n}+\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(2^{2k}-1)\,B_{2k}}{2k}\cdot\frac{1}{n^{2k+1}}+\frac{\pi}{24n}$$

$$+\frac{2}{\pi n}\log\frac{4}{\pi}-\frac{\pi}{24n}-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(2^{2k-1}-1)\,B_{2k}}{2k}\cdot\frac{1}{n^{2k+1}}$$

$$+4\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(2^{2k-1}-1)^2\,B_{2k}^2}{2k(2k)!}\left(\frac{\pi}{2}\right)^{2k-1}\cdot\frac{1}{n^{2k+1}}$$

$$\sim\frac{2}{\pi}\frac{\log n}{n}+\frac{2}{\pi}\left[\log\frac{8}{\pi}+\gamma\right]\frac{1}{n}+4\sum_{k=1}^{\infty}\frac{A_k^*}{n^{2k+1}} \qquad \text{as} \quad n=2m\to\infty, \tag{3.5}$$

where

$$A_k^* = \frac{(-1)^{k-1}(2^{2k-1}-1)^2\,B_{2k}^2}{2k(2k)!}\left(\frac{\pi}{2}\right)^{2k-1} \qquad \text{for} \quad k=1,2,3,\ldots$$

This completes the proof of Theorem 5.

Remark. It would be beneficial at this stage to examine the size of the coefficients appearing in the asymptotic expansion given by (3.5). Accordingly, if we put

$$A_0 := \frac{2}{\pi}\left[\log\frac{8}{\pi}+\gamma\right]$$

and

$$A_k := 4A_k^* \qquad \text{for} \quad k = 1, 2, 3, \ldots,$$

then (3.5) can be expressed in the form

$$\Delta_n \sim \frac{2}{\pi} \frac{\log n}{n} + \sum_{k=0}^{\infty} \frac{A_k}{n^{2k+1}} \qquad \text{as} \quad n = 2m \to \infty.$$

Approximate values for some of the coefficients A_k are listed in Table I.

In view of the size of the coefficients A_k, Δ_n can be most conveniently expressed in the form

$$\Delta_n = \frac{2}{\pi} \frac{\log n}{n} + \frac{2}{\pi}\left[\log \frac{8}{\pi} + \gamma\right] \frac{1}{n} + \frac{A_1}{n^3} + \frac{A_2}{n^5} + \alpha_n,$$

where $0 < \alpha_n < 0.0048/n^7$.

4. An Approach to the Proof of Theorem 7

It remains to prove Theorem 7. Accordingly, let n be any fixed but arbitrary even natural number. Put

$$D(\theta) = D_n(\theta) := \Delta_n(\cos \theta) \qquad \text{for} \quad 0 \leqslant \theta \leqslant \pi, \text{ where } x = \cos \theta. \quad (4.1)$$

Then we need to show that $D(\theta) \leqslant D(\pi/2)$ for $0 \leqslant \theta \leqslant \pi$. Our approach to this problem is as follows: given $j \in \{1, 2, 3, \ldots, m\}$, we shall define

$$R^j(\theta) = R_n^j(\theta) := D\left(\theta + \frac{\pi j}{n}\right) \qquad \text{for} \quad \frac{-\pi}{2n} \leqslant \theta \leqslant \frac{\pi}{2n}. \qquad (4.2)$$

TABLE I

k	A_k
0	0.9625
1	0.0436
2	−0.0088
3	0.0048
4	−0.0052
5	0.0096
6	−0.0268
7	0.1061
8	−0.5644

Furthermore, define

$$\begin{aligned} R^0(\theta) = R^0_n(\theta) &:= D(\theta+\pi) \qquad \text{for} \quad \frac{-\pi}{2n} \leqslant \theta < 0 \\ &= R^0_n(\theta) := D(\theta) \qquad \text{for} \quad 0 \leqslant \theta \leqslant \frac{\pi}{2n}. \end{aligned} \tag{4.3}$$

Then, to show that $D(\theta) \leqslant D(\pi/2)$ for $0 \leqslant \theta \leqslant \pi$, it suffices to establish that

(i) $R^j(\theta) < R^m(\theta)$ for $j = 0, 1, 2, \ldots, m-1$ and $-\pi/2n < \theta < \pi/2n$ and

(ii) $R^m(\theta) \leqslant R^m(0)$ for $-\pi/2n \leqslant \theta \leqslant \pi/2n$.

Our first task will be to obtain a suitable formula for $D(\theta)$. Accordingly, let $n = 2m$. From (4.1) and (2.4) we have

$$\begin{aligned} D(\theta) &= \frac{\cos^2 n\theta}{n^2} \sum_{k=1}^{n} \frac{(1-\cos\theta\cos\theta_k)}{|\cos\theta - \cos\theta_k|} \\ &= \frac{\cos^2 n\theta}{n^2} \sum_{k=1}^{n} E(\theta, \theta_k), \qquad \text{say.} \end{aligned} \tag{4.4}$$

Case A. $\theta_m < \theta < \theta_{m+1}$. For $k \in \{1, 2, 3, \ldots, m\}$, we combine $E(\theta, \theta_k)$ with $E(\theta, \theta_{n+1-k})$ to produce the formula

$$D(\theta) = \frac{2\sin^2\theta\cos^2 n\theta}{n^2} \sum_{k=1}^{m} \frac{\cos\theta_k}{\cos^2\theta_k - \cos^2\theta}.$$

Then

$$\begin{aligned} D(\theta) &= \frac{\sin\theta\cos^2 n\theta}{n^2} \sum_{k=1}^{m} \left(\frac{\sin\theta}{\cos\theta_k - \cos\theta} + \frac{\sin\theta}{\cos\theta_k + \cos\theta} \right) \\ &= \frac{\sin\theta\cos^2 n\theta}{2n^2} \sum_{k=1}^{m} \left(\cot\frac{\theta+\theta_k}{2} + \cot\frac{\theta-\theta_k}{2} + \tan\frac{\theta+\theta_k}{2} + \tan\frac{\theta-\theta_k}{2} \right). \end{aligned}$$

But $\cot A + \tan A = 2\operatorname{cosec} 2A$, so we deduce that

$$D(\theta) = \frac{\sin\theta\cos^2 n\theta}{n^2} \sum_{k=1}^{m} [\operatorname{cosec}(\theta+\theta_k) + \operatorname{cosec}(\theta-\theta_k)]. \tag{4.5}$$

Case B. $\theta_j < \theta < \theta_{j+1}$ for any $j \in \{1, 2, 3, \ldots, m-1\}$.

$$\begin{aligned} D(\theta) = {} & \frac{\sin\theta\cos^2 n\theta}{n^2} \sum_{k=1}^{j} [\operatorname{cosec}(\theta+\theta_k) + \operatorname{cosec}(\theta-\theta_k)] \\ & + \frac{\cos\theta\cos^2 n\theta}{n^2} \sum_{k=j+1}^{m} \tan\theta_k [\cot(\theta_k+\theta) + \cot(\theta_k-\theta)]. \end{aligned} \tag{4.6}$$

Case C. $0 \leqslant \theta < \theta_1$.

$$D(\theta) = \frac{\cos\theta \cos^2 n\theta}{n^2} \sum_{k=1}^{m} \tan\theta_k[\cot(\theta_k + \theta) + \cot(\theta_k - \theta)]. \tag{4.7}$$

Case D. $\theta = \theta_k$ for any $k \in \{1, 2, 3, ..., m\}$.

$$D(\theta_k) = 0. \tag{4.8}$$

Case E. $\theta_{m+1} < \theta \leqslant \pi$.

$$D(\theta) = D(\pi - \theta). \tag{4.9}$$

Thus a convenient formula for $D(\theta)$ is given by (4.5)–(4.9).

For notational convenience, now put $\theta_0 := 0$ and $\theta_{n+1} := \pi$. The following result is fundamental for our proof of Theorem 7.

THEOREM 8. *$D(\theta)$ is a piecewise trigonometric polynomial on $[0, \pi]$. More precisely, $D(\theta)$ is a trigonometric polynomial on $[\theta_j, \theta_{j+1}]$ for $j = 0, 1, 2, ..., n$. Furthermore, $D(\theta)$ has one and only one local maximum in $[\theta_j, \theta_{j+1}]$ for $j = 0, 1, 2, ..., n$.*

The proof of this theorem follows immediately from Theorem 6.

COROLLARY. *The restriction of $D(\theta)$ to $[\theta_m, \theta_{m+1}]$ attains its maximum value at $\theta = \pi/2$.*

Proof. Denote by $D^m(\theta)$ the restriction of $D(\theta)$ to $[\theta_m, \theta_{m+1}]$. From (4.5) and (4.8) we have

$$\begin{aligned} D^m(\theta) &= \frac{\sin\theta \cos^2 n\theta}{n^2} \sum_{k=1}^{m} [\operatorname{cosec}(\theta + \theta_k) + \operatorname{cosec}(\theta - \theta_k)] \\ &\qquad \text{for} \quad \theta_m < \theta < \theta_{m+1} \\ &= 0 \qquad \text{for} \quad \theta = \theta_m, \theta_{m+1}. \end{aligned}$$

Since $D^m(\theta) \geqslant 0$, $D^m(\theta)$ is symmetric about $\theta = \pi/2$ and $D^m(\theta)$ has one and only one maximum, it follows that this maximum is located at $\theta = \pi/2$.

From (4.2), (4.3), (4.5)–(4.9) we can obtain suitable formulae for the functions $R^j(\theta)$, for $j = 0, 1, 2, ..., n$. As the calculations are somewhat lengthy and quite detailed, we shall merely state the resulting formulae.

Accordingly, put $\gamma_j := \pi j/n$ and consider $-\pi/2n < \theta < \pi/2n$.

$$R^0(\theta) = \frac{\cos\theta \cos^2 n\theta}{n^2} \sum_{k=1}^{m} \tan\theta_k[\cot(\theta_k + \theta) + \cot(\theta_k - \theta)]. \tag{4.10}$$

If $1 \leqslant j < m/2$,

$$R^j(\theta) = \frac{\sin(\theta+\gamma_j)\cos^2 n\theta}{n^2} \sum_{k=1}^{2j} \operatorname{cosec}(\theta+\theta_k)$$
$$+ \frac{\cos(\theta+\gamma_j)\cos^2 n\theta}{n^2} \sum_{k=j+1}^{m-j} \tan\theta_k [\cot(\theta_{k-j}-\theta)$$
$$+ \cot(\theta_{k+j}+\theta)]$$
$$+ \frac{\cos(\theta+\gamma_j)\cos^2 n\theta}{n^2} \sum_{k=m-j+1}^{m} \tan\theta_k [\cot(\theta_{k-j}-\theta)$$
$$- \cot(\theta_{n+1-k-j}-\theta)]. \tag{4.11}$$

Now suppose m is even and $j = m/2$. Then $\gamma_j = \pi/4$ and

$$R^j(\theta) = \frac{\sin(\theta+\pi/4)\cos^2 n\theta}{n^2} \sum_{k=1}^{2j} \operatorname{cosec}(\theta+\theta_k)$$
$$+ \frac{\cos(\theta+\pi/4)\cos^2 n\theta}{n^2} \sum_{k=j+1}^{m} \tan\theta_k [\cot(\theta_{k-j}-\theta)$$
$$- \cot(\theta_{n+1-k-j}-\theta)]. \tag{4.12}$$

If $m/2 < j \leqslant m-1$,

$$R^j(\theta) = \frac{\sin(\theta+\gamma_j)\cos^2 n\theta}{n^2} \sum_{k=1}^{m} \operatorname{cosec}(\theta+\theta_k)$$
$$+ \frac{\sin(\theta+\gamma_j)\cos^2 n\theta}{n^2} \sum_{k=n+1-2j}^{m} \operatorname{cosec}(\theta_k-\theta)$$
$$+ \frac{\cos(\theta+\gamma_j)\cos^2 n\theta}{n^2} \sum_{k=j+1}^{m} \tan\theta_k [\cot(\theta_{k-j}-\theta)$$
$$- \cot(\theta_{n+1-k-j}-\theta)]. \tag{4.13}$$

$$R^m(\theta) = \frac{\cos\theta\cos^2 n\theta}{n^2} \sum_{k=1}^{m} [\operatorname{cosec}(\theta+\theta_k) + \operatorname{cosec}(\theta_k-\theta)]. \tag{4.14}$$

Also $R^j(-\pi/2n) = R^j(\pi/2n) = 0$ for $j = 0, 1, 2, \ldots, m$. (4.15)

The above formulae provide the cornerstone for the proof of Theorem 7.

5. The Proof of Theorem 7

It will suffice to show that, for any $\theta \in (-\pi/2n, \pi/2n)$,

$$R^m(\theta) > R^j(\theta) \qquad \text{for} \quad j = 0, 1, 2, \ldots, m-1.$$

Now from the definition of $R^m(\theta)$ and the corollary to Theorem 8, we see that $R^m(\theta)$ attains its maximum value at $\theta = 0$. The proof will then be completed once we have made the trivial observation that $R^m(0) = D(\pi/2)$.

In view of (4.10)–(4.13), it is clear that there are four separate cases to consider. Since these are somewhat repetitious, we shall consider one of them. Specifically, given $m/2 < j \leqslant m-1$ and $-\pi/2n < \theta < \pi/2n$, we shall show that

$$R^m(\theta) - R^j(\theta) > 0.$$

Now from (4.13) and (4.14) we have

$$\begin{aligned}
&R^m(\theta) - R^j(\theta)\\
&= \frac{[\cos\theta - \sin(\theta+\gamma_j)]\cos^2 n\theta}{n^2} \sum_{k=1}^{m} \operatorname{cosec}(\theta+\theta_k)\\
&\quad + \frac{[\cos\theta - \sin(\theta+\gamma_j)]\cos^2 n\theta}{n^2} \sum_{k=n+1-2j}^{m} \operatorname{cosec}(\theta_k-\theta)\\
&\quad + \frac{\cos\theta\cos^2 n\theta}{n^2} \sum_{k=1}^{n-2j} \operatorname{cosec}(\theta_k-\theta)\\
&\quad - \frac{\cos(\theta+\gamma_j)\cos^2 n\theta}{n^2} \sum_{k=j+1}^{m} \tan\theta_k\,[\cot(\theta_{k-j}-\theta) - \cot(\theta_{n+1-k-j}-\theta)]\\
&= \frac{[\cos\theta - \cos(\gamma_{m-j}-\theta)]\cos^2 n\theta}{n^2} \sum_{k=1}^{m} \operatorname{cosec}(\theta+\theta_k)\\
&\quad + \frac{[\cos\theta - \cos(\gamma_{m-j}-\theta)]\cos^2 n\theta}{n^2} \sum_{k=n+1-2j}^{m} \operatorname{cosec}(\theta_k-\theta)\\
&\quad + \frac{\cos\theta\cos^2 n\theta}{n^2} \sum_{k=j+1}^{m} [\operatorname{cosec}(\theta_{k-j}-\theta) + \operatorname{cosec}(\theta_{n+1-k-j}-\theta)]\\
&\quad - \frac{\cos(\theta+\gamma_j)\cos^2 n\theta}{n^2} \sum_{k=j+1}^{m} \tan\theta_k\,[\cot(\theta_{k-j}-\theta) - \cot(\theta_{n+1-k-j}-\theta)]\\
&= \left(2\sin\left(\frac{(m-j)\pi}{2n} - \theta\right)\sin\frac{(m-j)\pi}{2n}\cos^2 n\theta\Big/ n^2\right) \sum_{k=1}^{m} \operatorname{cosec}(\theta+\theta_k)\\
&\quad + \left(2\sin\left(\frac{(m-j)\pi}{2n} - \theta\right)\sin\frac{(m-j)\pi}{2n}\cos^2 n\theta\Big/ n^2\right) \sum_{k=n+1-2j}^{m} \operatorname{cosec}(\theta_k-\theta)\\
&\quad + \frac{\cos\theta\cos^2 n\theta}{n^2} \sum_{k=j+1}^{m} [\operatorname{cosec}(\theta_{k-j}-\theta) + \operatorname{cosec}(\theta_{n+1-k-j}-\theta)]\\
&\quad - \frac{\cos(\theta+\gamma_j)\cos^2 n\theta}{n^2} \sum_{k=j+1}^{m} \tan\theta_k\,[\cot(\theta_{k-j}-\theta) - \cot(\theta_{n+1-k-j}-\theta)].
\end{aligned} \tag{5.1}$$

Now the first two terms of (5.1) are strictly positive. Thus, to show that $R^m(\theta) - R^j(\theta) > 0$, it will suffice to show that for $k = j+1, j+2, \ldots, m$,

$$\cos\theta[\operatorname{cosec}(\theta_{k-j} - \theta) + \operatorname{cosec}(\theta_{n+1-k-j} - \theta)] > \cos(\theta + \gamma_j)\tan\theta_k[\cot(\theta_{k-j} - \theta) - \cot(\theta_{n+1-k-j} - \theta)]. \quad (5.2)$$

Now

$$\begin{aligned}\cos\theta[\operatorname{cosec}(\theta_{k-j} - \theta) &+ \operatorname{cosec}(\theta_{n+1-k-j} - \theta)] \\ &= \frac{\cos\theta[\sin(\theta_{k-j} - \theta) + \sin(\theta_{n+1-k-j} - \theta)]}{\sin(\theta_{k-j} - \theta)\sin(\theta_{n+1-k-j} - \theta)} \\ &= \frac{2\cos\theta\sin(\gamma_{m-j} - \theta)\cos\theta_{m+1-k}}{\sin(\theta_{k-j} - \theta)\sin(\theta_{n+1-k-j} - \theta)} \\ &= \frac{2\cos\theta\cos(\theta + \gamma_j)\sin\theta_k}{\sin(\theta_{k-j} - \theta)\sin(\theta_{n+1-k-j} - \theta)}. \end{aligned} \quad (5.3)$$

On the other hand,

$$\begin{aligned}&\cos(\theta + \gamma_j)\tan\theta_k[\cot(\theta_{k-j} - \theta) - \cot(\theta_{n+1-k-j} - \theta)] \\ &= \frac{\cos(\theta + \gamma_j)\sin\theta_k}{\cos\theta_k} \\ &\quad \times \left[\frac{\sin(\theta_{n+1-k-j} - \theta)\cos(\theta_{k-j} - \theta) - \cos(\theta_{n+1-k-j} - \theta)\sin(\theta_{k-j} - \theta)}{\sin(\theta_{k-j} - \theta)\sin(\theta_{n+1-k-j} - \theta)}\right] \\ &= \frac{\cos(\theta + \gamma_j)\sin\theta_k\sin 2\theta_{m+1-k}}{\cos\theta_k\sin(\theta_{k-j} - \theta)\sin(\theta_{n+1-k-j} - \theta)} \\ &= \frac{2\sin\theta_k\cos(\theta + \gamma_j)\sin\theta_k}{\sin(\theta_{k-j} - \theta)\sin(\theta_{n+1-k-j} - \theta)}. \end{aligned} \quad (5.4)$$

Comparing (5.3) with (5.4), we see that since $\cos\theta > \sin\theta_k$, so (5.2) is valid. We deduce that $R^m(\theta) > R^j(\theta)$ for $m/2 < j \leqslant m-1$. The other three cases may be treated in a similar fashion. This completes the proof of Theorem 7.

Remark. A more satisfactory result would be to establish that $R^{j+1}(\theta) > R^j(\theta)$ for $j \in \{0, 1, 2, \ldots, m-1\}$ and $-\pi/2n < \theta < \pi/2n$.

This would imply that the local maxima (of $D(\theta)$), which for $j = 1, 2, 3, \ldots, m-1$ are located to the right of the midpoint of $[\theta_j, \theta_{j+1}]$ are monotonically increasing as we proceed from either of the endpoints of $[0, \pi]$ to the centre of the interval. Contrast this phenomenon with the behaviour of the trigonometric Lebesgue function, whose local maxima are monotonically decreasing as we proceed from either of the endpoints of $[0, \pi]$ to the centre of the interval: see Luttmann and Rivlin [6], Brutman [2], and Günttner [5].

Indeed, Brutman's approach to the maximum problem for the trigonometric Lebesgue function provided us with a valuable clue as to how to prove Theorem 7, whereas Günttner's ideas have been helpful in establishing Lemma 1.

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